Inelastic Resonances and Castillejo-Dalitz-Dyson Singularities*

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By considering many coupled two-body-channel scattering amplitudes that possess a unique continuation in the angular-momentum variable l, we show: (1) that several methods of obtaining the single-channel N/D solution for the elastic amplitude with an inelasticity factor agree with the corresponding many-channel matrix N/D solution at high l; (2) that in order to preserve agreement when continuing to low angular momentum one must generally introduce Castillejo-Dalitz-Dyson (CDD) cuts or poles into the single-channel D function; (3) that the integral equations for N/D may be continued in l, automatically yielding the CDD cuts mentioned above. Finally, we introduce a distinction between elastic and inelastic resonance poles based on continuation in l and show that the latter will not appear in calculations that fail to include the requisite CDD singularities.

I. INTRODUCTION

THE incorporation of inelastic effects into N/D equations is an important problem of S-matrix theory. When such effects are incorporated in bootstrap calculations, the answers are generally quite different from the results obtained when inelasticity is ignored.[1]

In particular, inelastic effects may alter in an important way the position and width of a resonance which is bootstrapped on the basis of elastic unitarity.[1] Here we develop the idea that the very existence of the resonance pole may be a consequence of inelastic states. Any calculation in these cases, which neglects such states and considers only elastic effects, should not produce a resonance pole at all.

We will distinguish two types of resonance poles on the unphysical sheet reached by passing through the elastic cut. Elastic resonances are defined as those which migrate to the left-hand cut on this sheet as the angular momentum l becomes large.[2] Inelastic resonances are those which retreat through the right-hand inelastic cut in this limit. Resonances of the latter type will not be produced in any calculation which ignores inelasticity.

In order to calculate inelastic effects, the most direct approach would be the many-channel matrix N/D equations.[3] Alternatively, several authors[4–7] have proposed methods for introducing inelastic effects into the single-channel problem. In such problems an inelasticity coefficient must be given as input information in addition to the specification of left-hand cut singularities.

On the basis of models, Bander, Coulter, and Shaw,[8] Squires,[9] and Atkinson, Dietz, and Morgan[10] have recently shown that many-channel N/D and the single-channel methods with an inelasticity factor do not generally give the same answers. We demonstrate here, in a manner which is independent of detailed models, that the discrepancy between these various techniques arises as a consequence of the presence of inelastic resonance poles.

Assuming analyticity of the amplitudes in the angular-momentum variable l, we readily establish the relation between the various N/D procedures at high l where there are no resonances or bound-state poles. By continuation in l we find that the single-channel schemes with inelasticity depart from the many-channel result at the point where inelastic resonances first emerge from the inelastic cuts. At that point the analytically continued single-channel D function develops branch points on the physical sheet at the positions corresponding to the inelastic resonances on the second sheet. We then show that one may retrieve the analytically continued result for the amplitude by solving the single-channel problem with a newly defined D function which has Castillejo-Dalitz-Dyson (CDD)[11] poles that produce the inelastic resonance poles. Since the position of these resonances is then a part of the input data to the problem, these numbers cannot be computed by any single-channel calculation without CDD singularities.

In our discussion we make the requirement that the left-hand-cut contribution to the amplitude vanish for large energies and also for large angular momenta. This enables us to find solutions to the N/D equations for which the S matrix approaches one for the same limits. We also assume the left-hand-cut contribution is finite at all points along the left cut.

One can carry out a discussion similar to the one given here by considering analytic continuations in the

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[1] See, for example, F. Zachariasen and C. Zemach, Phys. Rev. **128**, 849 (1962).

[2] A general discussion of the motion of resonance poles with angular momentum is given in C. E. Jones, Ann. Phys. **31**, 481 (1965).

[3] See, for example, J. Bjorken, Phys. Rev. Letters **4**, 473 (1960).

[4] G. Frye and R. Warnock, Phys. Rev. **130**, 478 (1963).

[5] M. Froissart, Nuovo Cimento **22**, 191 (1961).

[6] J. Ball and W. Frazer, Phys. Rev. Letters **7**, 204 (1961).

[7] G. F. Chew, Phys. Rev. **129**, 2363 (1963); G. F. Chew and C. E. Jones, *ibid.* **135**, B208 (1964).

[8] M. Bander, P. Coulter, and G. Shaw, Phys. Rev. Letters **14**, 270 (1965).

[9] E. Squires, Nuovo Cimento **34**, 1751 (1964).

[10] D. Atkinson, K. Dietz, and D. Morgan, CERN (unpublished report).

[11] L. Castillejo, R. Dalitz, and F. Dyson, Phys. Rev. **101**, 453 (1956).

coupling constants. The angular-momentum variable, however, has the advantage of constituting a single strength parameter, which is well defined and relevant in all problems.

II. SOLUTIONS FOR LARGE ANGULAR MOMENTUM

We shall now consider several methods for solving a single-channel problem for the elastic amplitude A incorporating inelastic effects. Because the force vanishes and the S matrix approaches one for large l, we shall be able to determine what (if any) CDD singularities must be included in the solution.

The elastic amplitude A can be written

$$A=(2i\rho)^{-1}(S-1)=(2i\rho)^{-1}(\eta e^{2i\delta}-1)=\mathcal{R}e^{i\phi}, \quad (1)$$

where η, δ, $\mathcal{R}$, ϕ are functions of the angular momentum l and energy. They are real for real angular momenta and energy above the elastic threshold s_0. The function η is one below the inelastic threshold s_1, and above it satisfies $0\leq\eta\leq 1$. The function ρ is the phase-space factor; δ and ϕ are normalized to zero at infinite energy.

The first method we consider is that of Frye and Warnock.[4] In this method the amplitude is written as N/D, where D is required to have the following properties:

(a) $D(s)$ has the phase $-\delta(s)$ on the right-hand cut;
(b) $D(s)\to 1$ as $s\to\infty$;
(c) the zeros of $D(s)$ are in one-to-one correspondence with the bound-state poles of $A(s)$.

In the large-l limit the phase δ must be tending to zero for all energy in order that the amplitude approach zero. For sufficiently large l, then, $\delta(s_0)=0$. Also in this limit there are no bound-state poles. A $D(s)$ consistent with (a), (b), and (c) can then be defined as

$$D(s)=\exp\left(-\frac{1}{\pi}\int_{s_0}^{\infty}\frac{\delta(s')}{s'-s}ds'\right). \quad (2)$$

Under our assumptions about the asymptotic behavior of the left-hand-cut contribution, the equations for $N(s)$ [see Eq. (8)] which result from this form of D are soluble. The solution is also seen to have the desired property that $A(s)\to 0$ as $s\to\infty$. This form of $D(s)$ cannot be modified by adding CDD poles or square-root singularities of the type discussed in Ref. 12 without violating properties (b) and (c).

[12] J. B. Hartle and C. E. Jones, Phys. Rev. Letters **14**, 801 (1965).

We remark that a similar result also holds for the method of Froissart[5] where

$$D(s)=\exp\left(-\frac{1}{\pi}\int_{s_0}^{\infty}\frac{\alpha(s')}{s'-s}ds'\right),$$
$$\alpha(s)=\delta(s)+\frac{P}{2\pi}\int_{s_1}^{\infty}\frac{\ln\eta(s')}{s'-s}ds', \quad (3)$$

and in the method of Ball and Frazer.[6]

One cannot always dispense with CDD singularities at large l. In particular, let us consider solving for A by replacing assumption (a) by the assumption (a′) that D carries the entire right-hand cut of A, the so-called R method. We shall consider the question of whether CDD poles can be excluded in this case at large angular momentum. The N/D equations which result from (a′), (b), (c), including possible CDD poles are

$$N(s)=B(s)+\sum_{i=1}^{n}\gamma_i\left[\frac{B(s)-B(p_i)+A(p_i)}{s-p_i}\right]+\frac{1}{\pi}\int_{s_0}^{\infty}ds'\frac{B(s')-B(s)}{s'-s}R(s')\rho(s')N(s'), \quad (4)$$

$$D(s)=1+\sum_i\frac{\gamma_i}{s-p_i}-\frac{1}{\pi}\int_{s_0}^{\infty}\frac{ds'}{s'-s}R(s')\rho(s')N(s'). \quad (5)$$

Here $B(s)$ is the left-hand-cut contribution to the amplitude; $R(s)$ is the ratio of total to elastic partial cross sections; p_i, γ_i, and $A(p_i)$ specify the CDD poles and together with B and R provide the input information to the problem.

At large l where bound states are absent the D function which results from (5) must have the form

$$D(s)=(s-s_0)^n\prod_{i=1}^{n}(s-p_i)^{-1}\exp\left(-\frac{1}{\pi}\int_{s_0}^{\infty}\frac{\phi(s')}{s'-s}ds'\right). \quad (6)$$

Since D has no poles or zeros at $s=s_0$ we must have

$$\phi(s_0)=-n\pi. \quad (7)$$

We now show that with this method $\phi(s_0)$ may be nonzero at large l and one must, therefore, generally include CDD poles at large l. For example, consider a problem in which $B(s)=0$. If there were no CDD poles Eq. (5) would imply $N(s)=0$ and therefore $A(s)=0$.[13] Now, it is easy to conceive of models in which the direct force is zero, but for which the amplitude does not vanish. For example, the model studied in Ref. 8 is of this type. For such problems CDD singularities are clearly required.

In summary we relate the single-channel methods discussed here to the many-channel matrix N/D solu-

[13] We are indebted to Professor R. Blankenbecler for calling our attention to the failure of the R method in this case.

tion. The matrix N/D equations without CDD poles and with a driving term $B_{ij}(s)$, which vanishes at infinite energies, are easily seen to have solutions with the desired properties ($A_{ij}\to 0$ or $S_{ij}\to 1$ as $l\to\infty$ or $s\to\infty$). We can compute from this result the inelasticity coefficients η and R. If these coefficients are now introduced into the single-channel methods discussed above, we get agreement with A_{11} without adding CDD poles for large angular momentum except in the case of the method involving R where generally CDD poles must be added.[13a]

III. CONTINUATION IN THE ANGULAR MOMENTUM

Now we turn to the problem of whether CDD singularities are required at lower values of the angular momentum. We will consider in detail the Frye-Warnock method. Assuming the form of D given by Eq. (2), the integral equation for $R\,N(s)$ takes the form

$$\frac{2\eta(s)}{1+\eta(s)}\mathrm{Re}N(s)=\bar{B}(s)+\frac{1}{\pi}\int_{s_1}^{\infty}\frac{\bar{B}(s')-\bar{B}(s)}{s'-s}\frac{2\rho(s')}{1+\eta(s')}\mathrm{Re}N(s')ds',\quad(8)$$

where

$$\bar{B}(s)=B(s)+\frac{P}{\pi}\int_{s_1}^{\infty}\frac{1-\eta(s')}{2\rho(s')(s'-s)}ds',\quad(9)$$

and then

$$D(s)=1-\frac{1}{\pi}\int_{s_0}^{\infty}\frac{2\rho(s')\mathrm{Re}N(s')}{(s'-s)(1+\eta(s'))}ds',$$
$$N(s)=\mathrm{Re}N(s)+\mathrm{Re}D(s)(1-\eta)/2\rho.\quad(10)$$

We shall demonstrate how to continue the solution of these equations obtained for large l where there are no CDD singularities to lower values of l. By CDD singularities we mean poles or cuts of N and D which do not appear in A. If CDD singularities are required at these lower values, they will emerge automatically from the cuts of D during the continuation process.

We shall assume that the singularities of A are confined to the left- and right-hand cuts and the bound-state poles. No CDD cuts[12] can therefore develop out of the left-hand cut as l is decreased, since $N(s)$ contains all of the left-hand cut and there would be nothing in D to cancel such CDD cuts. CDD poles cannot come from the left-hand cut because of the assumed finiteness of $B(s)$.

No CDD singularities can emerge through the elastic cut. Suppose that there were a CDD singularity in D which retreated through the elastic cut as l was increased. Since, in the elastic region, S can be written

$$S(s)=D^*(s)/D(s),\quad(11)$$

it follows that S must have a zero (if D has a CDD pole) or a cut (if D has a CDD cut) on the unphysical sheet reached by going through the elastic cut. The continuation of elastic unitarity, $S(s)S^*(s)=1$, then implies that S must have a corresponding pole or cut on the physical sheet. This violates our assumption that the only singularities of S on the physical sheet for large l are the left- and right-hand cuts. No CDD singularities, then, can come from the elastic cut. Indeed, for a problem with no inelasticity ($\eta=1$) this result shows that there can be no CDD singularities at any value of the angular momentum if solutions are to satisfy our boundary conditions as $l\to\infty$ and be analytic functions of l. We have already seen that no CDD poles are required in the matrix N/D equations in order to yield the right asymptotic behavior in l and s. This has also been shown to be the case for the N/D equations used in the strip approximation.[14]

The only remaining source of CDD singularities is the inelastic cut. As Bander, Coulter, and Shaw[8] have emphasized, if $\eta>0$, Eq. (8) is of Fredholm type and no singularity can occur in the solution. Only when η assumes the value zero on the real axis does the equation cease to be of Fredholm type and the possibility of a singularity occur.

The function η vanishes on the real axis when a resonance pole emerges from the inelastic cut onto the unphysical sheet reached by going through the elastic cut (see Ref. 12). This type of resonance we have called an inelastic resonance. The continuation of elastic unitarity requires that S vanish at the corresponding point on the physical sheet. We shall assume that the resonance pole is simple.

On the real axis $\eta(s)$ can be written as

$$\eta(s)=[S(s)S^*(s)]^{1/2},\quad(12)$$

where $\eta(s)$ is real for real s. If the position of the zero of S on the upper-half physical sheet is denoted by s_v, the form of s is $S(s)=(s_v-s)\tilde{S}(s)$ where $\tilde{S}(s)$ is nonzero at s_v. Now the analytic continuation of $S^*(s)$ will have a zero at s_v^* and in general, will be nonzero at s_v. Otherwise at large l when s_v has retreated through the inelastic cut a pole would appear on the physical sheet in violation of our assumptions about the singularities of S in this limit. Thus $\eta(s)$ may be written

$$\eta(s)=[(s_v-s)(s_v^*-s)]^{1/2}\tilde{\eta}(s),\quad(13)$$

where $\tilde{\eta}(s)$ is regular at s_v and s_v^*. The function $\eta(s)$ is real above s_1 and has two complex-conjugate square-root branch points which cross the real axis as an inelastic resonance emerges onto the unphysical sheet.

We assume this form for the input inelasticity $\eta(s)$ and show that the solutions to Eqs. (8), (9), and (10) as l is decreased can be continued through the value at

[13a] The method involving R will be more carefully discussed in a forthcoming paper: J. B. Hartle and C. E. Jones (to be published).

[14] C. E. Jones (unpublished).

which s_v crosses the real axis. Let us define

$$\mathrm{Re}N(s)=\frac{1+\eta(s)}{2\eta(s)}\mathfrak{N}(s), \quad (14)$$

where $\mathfrak{N}(s)$ then satisfies

$$\mathfrak{N}(s)=\bar{B}(s)+\frac{1}{\pi}\int_{s0}^{\infty}\frac{\bar{B}(s')-\bar{B}(s)}{s'-s}\frac{\rho(s')}{\eta(s')}\mathfrak{N}(s'). \quad (15)$$

As l is continued through the value at which s_v crosses the real axis the solutions to Eq. (15) are finite in s. To see this we write

$$\frac{\bar{B}(s')-\bar{B}(s)}{s'-s}\frac{\rho(s')}{\eta(s')}=K(s,s')[(s'-s_v)(s'-s_v^*)]^{-1/2}. \quad (16)$$

The $K(s,s')$ is then always finite in s and s'. This kernel in turn can be written

$$K(s,s')[(s'-s_v)(s'-s_v^*)]^{-1/2}=P(s,s')+Q(s,s'), \quad (17)$$

where $P(s,s')$ is of finite rank and $Q(s,s')$ is of Fredholm type. We can take

$$P(s,s')=\frac{K(s,s_v)}{s_v-s_v^*}\left(\frac{s'-s_v^*}{s'-s_v}\right)^{1/2}-\frac{K(s,s_v^*)}{s_v-s_v^*}\left(\frac{s'-s_v}{s'-s_v^*}\right)^{1/2}, \quad (18)$$

$$Q(s,s')=\frac{K(s,s')-K(s,s_v)}{s_v-s_v^*}\left(\frac{s'-s_v^*}{s'-s_v}\right)^{1/2}-\frac{K(s,s')-K(s,s_v^*)}{s_v-s_v^*}\left(\frac{s'-s_v}{s'-s_v^*}\right)^{1/2}. \quad (19)$$

The solution to Eq. (15) can then be written as

$$\mathfrak{N}(s)=\int_{s0}^{\infty}R(s,s')\bar{B}(s')ds', \quad (20)$$

where R is the resolvent given in terms of P and Q by

$$R=[I-(I-P)^{-1}Q]^{-1}(I-P)^{-1}. \quad (21)$$

The operator $(I-P)^{-1}$ can be computed explicitly and remains well defined as s crosses the real axis, producing no singularities. A theorem of Tiktopoulos[15] then allows us to conclude that for such a kernel the solution can be analytically continued through the point where s_v crosses the real axis.

It is not necessary to consider this analytic continuation explicitly in order to determine what CDD singularities are required at low values of l. If η is assumed to have the form of Eq. (13), then it follows from Eq. (12) that S has a simple zero at s_v and s_v^* on the physical sheet. Since S can be written as $S=\eta(s)D^*(s)/D(s)$, it then follows that $D(s)$ is of the form

$$D(s)=[(s_v^*-s)(s_v-s)]^{-1/2}\tilde{D}(s), \quad (22)$$

[15] G. Tiktopoulos, Phys. Rev. **133**, B1231 (1964).

where D is regular at s_v and s_v^*. (D^* cannot vanish since this would imply that D vanishes on the physical sheet at large l in violation of our assumption.) Thus, D has CDD branch points at s_v and s_v^*; N can be written as $(\eta D^*-D)/2i\rho$ and therefore contains this CDD cut in a multiplicative way.

The representation for $D(s)$ corresponding to Eq. (2) can be found by examining the behavior of δ as the zero of S emerges onto the physical sheet. Since $\delta(s)=(1/4i)\ln[S(s)/S^*(s)]$, we can write

$$\delta(s)=(1/4i)\ln[(s_v^*-s)(s_v-s)]+\tilde{\delta}(s). \quad (23)$$

When s_v crosses the real axis the logarithmic branch points of δ distort the contour in the integral of Eq. (2). The integral over the first term in Eq. (23) gives rise to a multiplicative factor in D of the form $(s-s_p)\times[(s_v-s)(s_v^*-s)]^{-1/2}$, where s_p is the point at which the branch cuts of δ cross the real axis. The phase δ on the real axis will have a discontinuity of π at s_p so we will write it as $\delta(s,p)$. Then D takes the form

$$D(s)=\frac{s-s_p}{[(s_v-s)(s_v^*-s)]^{1/2}}\exp\left(-\frac{1}{\pi}\int_{s0}^{\infty}\frac{\delta(s',p)}{s'-s}ds'\right). \quad (24)$$

There is no zero of D at s_p since $\delta(s,p)$ has a discontinuity there. The point s_p can be taken to be any point on the right-hand cut.

The D function represented in Eq. (24) is completely determined as the analytic continuation of the D at large l with the given information about η contained in Eq. (18) and satisfies restrictions (a), (b), and (c) of Sec. II. We can obtain a form of D which is more convenient for calculations at low l by relaxing requirement (c) and allowing D to have zeros which do not correspond to bound states. We define

$$\begin{aligned}\bar{D}(s)&=\{(s-s_z)/[(s_v-s)(s_v^*-s)]^{1/2}\}D(s)\\ \bar{N}(s)&=\{(s-s_z)/[(s_v-s)(s_v^*-s)]^{1/2}\}N(s),\end{aligned} \quad (25)$$

where s_z is arbitrary but real.

The new function $\bar{D}$ has two CDD poles at s_v and s_v^*. Since s_z is arbitrary, there is one linear relation between the residues of these poles. If this relation is given, together with η and the position of the resonance s_v, we can write a set of equations like Eqs. (8), (9), and (10) now including the CDD poles which completely determine $A(s)$.

The situation is simpler if the angular momentum is decreased enough so that the resonance moves on to the physical sheet at s_0 as a bound state. Then s_v and s_v^* move onto the real axis, one to the position of the bound state s_B and the other to a point s_C. During this continuation a simple zero emerges from the exponential factor and moves to s_B giving N and D the form.

$$\begin{aligned}D(s)&=[(s_B-s)/(s_C-s)]^{1/2}\tilde{D}(s),\\ N(s)&=[(s_B-s)(s_C-s)]^{-1/2}\tilde{N}(s),\end{aligned} \quad (26)$$

where $\tilde{D}(s)$ and $\tilde{N}(s)$ are regular at s_B and s_C. By multiplying both factors by $[(s_C-s)/(s_B-s)]^{1/2}$, we arrive at functions N and D given by

$$\bar{N}(s)=\tilde{N}(s)(s_B-s)^{-1},\quad \bar{D}(s)=\tilde{D}(s). \tag{26a}$$

An integral equation analogous to Eq. (8) can now be derived for Re $\bar{N}$

$$\frac{2\eta(s)}{1+\eta(s)}\mathrm{Re}\bar{N}(s)=\bar{B}(s)+\frac{\bar{D}(s_B)\Gamma}{s-s_B}+\frac{1}{\pi}\int_{s_0}^{\infty}ds'\frac{\bar{B}(s')-\bar{B}(s)}{s'-s}\frac{2\rho(s')\mathrm{Re}\bar{N}(s')}{1+\eta(s')}, \tag{27}$$

where Γ is the residue of the bound-state pole in $A(s)$. The functions $\bar{D}(s)$ and $\bar{B}(s)$ are still given by Eqs. (9) and (10). If we evaluate Eq. (10) at $s=s_B$ and insert it in Eq. (27), we find

$$\frac{2\eta(s)}{1+\eta(s)}\mathrm{Re}\bar{N}(s)=\bar{B}(s)+\frac{\Gamma}{s-s_B}+\frac{1}{\pi}\int_{s_0}^{\infty}ds'\times\left[\frac{\bar{B}(s')-\bar{B}(s)}{s'-s}-\frac{\Gamma}{(s-s_B)(s'-s_B)}\right]\times\frac{2\rho(s')\mathrm{Re}\bar{N}(s')}{1+\eta(s')}. \tag{28}$$

If the position and residue of the bound-state pole in the amplitude are given, Eq. (28) allows us to calculate $\mathrm{Re}\bar{N}(s)$ and with the aid of Eq. (10) the amplitude $A(s)$.

Equation (28) will give us the correct solution at low values of l, but we see that information in addition to η and $B(s)$ must be included as input to the problem.

IV. CONCLUSION

We have illustrated here the use of analytic continuation in l as a tool for selecting unambiguously the desired single-channel N/D solution at low l, when inelastic effects are included. At large angular momentum, where there are no bound states, the solution with the requisite boundary conditions was easy to identify as one having no CDD singularities. We have seen that the integral equations could be continued to low values of l so that the amplitude is completely given, in principle, by a knowledge of η and the left-hand-cut contribution.

If in the continuation to lower values of l the amplitude develops inelastic resonances, η was shown to have square-root zeros crossing the real axis. The analytically continued D function then develops square-root branch points at the position of these resonances.

These branch points could be replaced by poles if D were redefined, but the resonance poles could not be expected to appear in the single-channel calculation unless the appropriate CDD singularities were put in by hand.

Elastic resonances, on the other hand, which emerge from the left-hand cut on the second sheet of the amplitude may well be present in a calculation where inelasticity is ignored.

This result suggests that a re-examination of elastic bootstrap calculations might be important to decide when there is a hope that the resonance being calculated is elastic. For cases of inelastic resonances, it appears that a bootstrap calculation based on elastic unitarity is inaccurate and misleading.[16]

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[16] See L. F. Cook and C. E. Jones (to be published).